JOURNAL OF ALGEBRA 130, 311–327 (1990)

Primitive Localizations of an Enveloping Algebra

L. DELVAUX, E. NAUWELAERTS, A. I. OOMS, AND P. WAUTERS

Department of Mathematics, L.U.C. and E.H.L., University of Limburg, 3610 Diepenbeek, Belgium

Communicated by Nathan Jacobson

Received May 30, 1988

We prove a number of sufficient conditions for a localization of an enveloping algebra at a multiplicatively closed set of semi-invariants to be a primitive ring. In case of a central localization these conditions are also necessary.

We fix the following notations and terminology. $U(L)$ denotes the universal enveloping algebra of a nonzero finite-dimensional Lie algebra L over a field k of characteristic zero. $Z(U(L))$ is the center of $U(L)$ and $D(L)$ is the division ring of quotients of $U(L)$ with center $Z(D(L))$. For each $\lambda \in L^*$, let $D(L)_\lambda = \{u \in D(L) \mid \forall x \in L \ \mathrm{ad}\, x(u) = \lambda(x)\, u\}$. These elements are called the semi-invariants of $D(L)$ with weight λ. Put $U(L)_\lambda = U(L) \cap D(L)_\lambda$. Let $\Lambda(L) = \{\lambda \in L^* \mid U(L)_\lambda \neq 0\}$ and $\Lambda_D(L) = \{\lambda \in L^* \mid D(L)_\lambda \neq 0\}$. Clearly $D(L)_\lambda D(L)_\mu \subset D(L)_{\lambda+\mu}$ for all λ, $\mu \in \Lambda_D(L)$. Moreover the sum of the $D(L)_\lambda$ is direct and is denoted by $Sz(D(L))$, the semicenter of $D(L)$. Similarly $Sz(U(L)) = \bigoplus_{\lambda \in \Lambda(L)} U(L)_\lambda$ is called the semicenter of $U(L)$, which is a commutative factorial domain [11, 13]. Note that $Sz(U(L))$ is a $\Lambda(L)$-graded ring. The set of nonzero semi-invariants of $U(L)$ is denoted by E. $\Lambda(L)$ is a commutative cancellative monoid and $\Lambda_D(L)$ turns out to be the group generated by $\Lambda(L)$ [4, p. 239] and $\Lambda_D(L)$ is a finitely generated torsion-free abelian group [17, Theorem 1.3].

By $\Lambda^u(L)$ we denote the group of units of $\Lambda(L)$, i.e., those $\lambda \in \Lambda(L)$ such that $-\lambda \in \Lambda(L)$. $\Lambda^u(L)$ is a finitely generated torsion-free abelian group, being a subgroup of $\Lambda_D(L)$. The subalgebra $\bigoplus_{\lambda \in \Lambda^u(L)} U(L)_\lambda$ of $Sz(U(L))$ is denoted by $Sz^u(U(L))$. This is a $\Lambda^u(L)$-graded ring with $(Sz^u(U(L)))_0 = U(L)_0 = Z(U(L))$.

LEMMA 1. *Let $\mu_1, ..., \mu_t$ be a basis of $\Lambda^u(L)$ and choose $0 \neq v_i \in U(L)_{\mu_i}$. Let S be the multiplicatively closed set of nonzero homogeneous elements of $Sz^u(U(L))$, i.e., the nonzero semi-invariants in $Sz^u(U(L))$. Then*

0021-8693/90 $3.00

(a) *$Sz^u(U(L))_S = Sz^u(U(L))_{Z(U(L))} = Q(Z(U(L)))[v_1, \ldots, v_t, v_1^{-1}, \ldots, v_t^{-1}]$ which is isomorphic to the group algebra $Q(Z(U(L)))[\Lambda^u(L)]$ of $\Lambda^u(L)$ over the quotient field $Q(Z(U(L)))$,*

(b) *$Sz^u(U(L))$ and the polynomial subring $Z(U(L))[v_1, \ldots, v_t]$ have the same quotient field. However, if $\Lambda^u(L) \neq \{0\}$ then $Sz^u(U(L))$ is not a polynomial algebra over $Z(U(L))$.*

Proof. (a) By $Sz^u(U(L))_{Z(U(L))}$ we mean the localization of $Sz^u(U(L))$ at the nonzero elements of $Z(U(L))$.

Let $s \in S$. So $s \in U(L)_\lambda$ for some $\lambda \in \Lambda^u(L)$. Thus there exists $0 \neq t \in S \cap U(L)_{-\lambda}$. Hence $0 \neq st \in Z(U(L))$. This shows that $Sz^u(U(L))_{Z(U(L))} = Sz^u(U(L))_S$ and its part of degree zero equals $Q(Z(U(L))$. Moreover, by construction $Sz^u(U(L))_S$ is a graded field, i.e., every nonzero homogeneous element is a unit. Since $\Lambda^u(L)$ is a finitely generated torsion-free abelian group it follows that $Sz^u(U(L))_S$ is isomorphic to the group algebra $Q(Z(U(L)))[\Lambda^u(L)]$ (see also [1, p. 87]. In particular,

$$Sz^u(U(L))_S = Q(Z(U(L)))[v_1, \ldots, v_t, v_1^{-1}, \ldots, v_t^{-1}],$$

(b) The first part follows from (*a*). On the oher hand, if $\Lambda^u(L) \neq \{0\}$ then we can find $s, t \in Sz^u(U(L)) \backslash Z(U(L))$ such that $st \in Z(U(L))$ (see (a)), showing that $Sz^u(U(L))$ is not a polynomial algebra over $Z(U(L))$. ∎

PROPOSITION 2.

(1) *$Sz(U(L)) = Sz^u(U(L)) \oplus J$ $(*)$, where $J = \oplus_{\lambda \notin \Lambda^u(L)} U(L)_\lambda$ is a prime ideal of $Sz(U(L))$.*

(2) *Let u, v be nonzero elements of $U(L)$. If uv and v belong to $Sz^u(U(L))$ then so does u.*

(3) *$Sz^u(U(L))$ is a factorial domain.*

Proof. (1) Let $u \in U(L)_\lambda$, $v \in U(L)_\mu$ where $\lambda \in \Lambda(L)$ and $\mu \in \Lambda(L) \backslash \Lambda^u(L)$. Then $uv \in U(L)_{\lambda+\mu}$ and $\lambda + \mu \notin \Lambda^u(L)$. Hence J is an ideal of $Sz(U(L))$. It is also prime, since $Sz(U(L))/J$ is isomorphic to the integral domain $Sz^u(U(L))$.

(2) Because uv and v belong to $Sz(U(L))$ so does u [11, Lemma 2]. Therefore there are $u_1 \in Sz^u(U(L))$ and $u_2 \in J$ such that $u = u_1 + u_2$. Then $uv = u_1 v + u_2 v$ is the unique decomposition of uv according to $(*)$ since $u_1 v \in Sz^u(U(L))$ and $u_2 v \in J$. But $uv \in Sz^u(U(L))$ by assumption. So $u_2 v = 0$ and hence also $u_2 = 0$. Consequently, $u = u_1 \in Sz^u(U(L))$.

(3) First we observe that if u, v are nonzero semi-invariants of $U(L)$ with weights $\lambda, \mu \in \Lambda(L)$ such that $uv \in Sz^u(U(L))$ then both

$u, v \in Sz^u(U(L))$. (Indeed, $uv \in U(L)_{\lambda+\mu}$ so $\lambda+\mu \in \Lambda^u(L)$ and thus $\lambda, \mu \in \Lambda^u(L)$). Next, we consider $Sz^u(U(L))$ as a $\Lambda^u(L)$-graded domain. Let u be a nonzero nonunit homogeneous element of $Sz^u(U(L))$, i.e., $u \in U(L)_\lambda \backslash k$ for some $\lambda \in \Lambda^u(L)$. By [4, Proposition 1.5), u has a unique decomposition $u = u_1 \cdots u_q$ into irreducible semi-invariants u_i of $Sz(U(L))$. By the remark, $u_1, ..., u_q \in Sz^u(U(L))$. Each u_i is prime in $Sz^u(U(L))$. Indeed, suppose u_i divides vw in $Sz^u(U(L))$ where v, w are nonzero elements of $Sz^u(U(L))$. In the UFD $Sz(U(L))$, u_i is prime and divides vw. Therefore u_i divides v (or w) in $Sz(U(L))$, i.e., $v = yu_i$ for some nonzero $y \in Sz(U(L))$. Now, yu_i and u_i belong to $Sz^u(U(L))$. By (2), this implies that $y \in Sz^u(U(L))$ and thus u_i divides v in $Sz^u(U(L))$. This shows that $Sz^u(U(L))$ is a graded UFD. Because the localization $Sz^u(U(L))_S$ is a UFD (see (a) of Lemma 1), we may conclude from [2, Theorem 4.4] that $Sz^u(U(L))$ is a UFD. ▮

Later on (see Remarks 15), we will give an example where $Z(U(L))$ itself is not factorial.

A ring R is said to be a Formanek ring if every nonzero ideal I of R has a nonzero intersection with the center of R.

PROPOSITION 3. *$U(L)$ is a Formanek ring if and only if $\Lambda(L)$ is a group (i.e., $\Lambda(L) = \Lambda^u(L) = \Lambda_D(L)$).*

Proof. It is shown in [12, 14] that every nonzero ideal of $U(L)$ intersects E, the set of nonzero semi-invariants of $U(L)$. Suppose that $U(L)$ is a Formanek ring. Take $\lambda \in \Lambda(L)$ and $0 \neq u \in U(L)_\lambda$. Then $U(L)u$ is a nonzero two-sided ideal of $U(L)$, hence $0 \neq U(L)u \cap Z(U(L))$. Therefore $vu \in Z(U(L))$ for some nonzero $v \in U(L)$. Clearly $v \in U(L)_{-\lambda}$ and thus $\lambda \in \Lambda^u(L)$, showing that $\Lambda(L)$ coincides with the group $\Lambda^u(L)$.

Conversely, suppose that $\Lambda(L)$ is a group. Let I be a nonzero ideal of $U(L)$. I contains a nonzero semi-invariant u of $U(L)$ with weight λ. Then, $-\lambda \in \Lambda(L)$. Hence there is a nonzero $v \in U(L)_{-\lambda}$. Consequently, $0 \neq uv \in I \cap Z(U(L))$. ▮

Note that $U(L)$ is a Formanek ring if and only if the central localization $U(L)_{Z(U(L))}$ is a simple ring. Further on, we will determine when $U(L)_{Z(U(L))}$ is a primitive ring.

In analogy with the commutative case, we say that a multiplicatively closed subset S of nonzero semi-invariants of $U(L)$ is saturated whenever $uv \in S$, where $u, v \in E$, implies that $u, v \in S$. Note that the condition $u, v \in E$ may be weakened to $u, v \in U(L)$ because $uv \in S \subset E$ implies that $u, v \in E$ [4, p. 327]. In particular, S contains k^*. If S is an arbitrary multiplicatively closed ubset of E, put $\mathrm{Sat}(S) = \{u \in E \mid uu' \in S \text{ for some } u' \in E\}$. It is readily checked that $\mathrm{Sat}(S)$ is saturated and multiplicatively closed and it is the smallest among these subsets of E containing S. We also have $U(L)_S = U(L)_{\mathrm{Sat}(S)}$, for if $t \in \mathrm{Sat}(S)$, then $tu = s \in S$ for some $u \in E$. In particular, $t^{-1} = s^{-1}u \in U(L)_S$. Moreover

$$(U(L)_S)^* = \{s^{-1}t \mid s \in S, t \in \mathrm{Sat}(S)\}$$

($(U(L)_S)^*$ denotes the group of units of $U(L)_S$). Also

$$\mathrm{Sat}(Z(U(L))\backslash\{0\}) = \bigcup_{\lambda \in \Lambda^u(L)} (U(L)_\lambda)\backslash\{0\}.$$

Finally, it is immediate that $U(L)_S = U(L)_{S'}$, if and only if $\mathrm{Sat}(S) = \mathrm{Sat}(S')$.

A saturated multiplicatively closed subset S of E is said to be weakly additively closed if s, $t \in S \cap U(L)_\lambda$ for some $\lambda \in \Lambda(L)$ implies that $s + t \in S \cup \{0\}$. In this case

$$Q(S) = \{st^{-1} \mid s, t \in S \cap U(L)_\lambda \text{ for some } \lambda \in \Lambda(L)\} \cup \{0\}$$

is clearly a subfield of $Z(D(L))$.

Conversely, if $Q(S)$ is a field and S is saturated, then S is weakly additively closed, for let s, $s' \in S \cap U(L)_\lambda$ with $s + s' \neq 0$; since $s^{-1}(s + s') = 1 + s^{-1}s' \in Q(S)$ we may write $s^{-1}(s + s') = t^{-1}t'$ where t, $t' \in S \cap U(L)_{\lambda'}$ for some $\lambda' \in \Lambda(L)$. Thus $t(s + s') = st' \in S$, hence $s + s' \in S$. In particular, if T is a saturated multiplicatively closed subset of E and if $Z(D(L)) = Q(T)$ then T is weakly additively closed. Furthermore, if S is contained in $Z(U(L))$, the condition that S is weakly additively closed means precisely that $S \cup \{0\}$ is a subalgebra of $Z(U(L))$ and $Q(S)$ is in that case the field of fractions of S.

PROPOSITION 4. *Let T be a saturated multiplicatively closed subset of E such that each nonzero element of $Z(D(L))$ is a quotient of two elements of T, i.e., $Z(D(L)) = Q(T)$.*

Denote by A the subalgebra of $Sz(U(L))$ generated by T. Let $u_1, ..., u_r$ be nonassociated irreducible semi-invariants not contained in T. Then $u_1, ..., u_r$ are algebraically independent over A.

Proof. Let t, $t_1 \in T$. If $(m_1, ..., m_r) \neq (n_1, ..., n_r)$ where all $m_i, n_i \in \mathbb{N}$, then the semi-invariants

$$\alpha = tu_1^{m_1} \cdots u_r^{m_r} \qquad \text{and} \qquad \alpha_1 = t_1 u_1^{n_1} \cdots u_r^{n_r}$$

have a different weight [For if not, then $\alpha\alpha_1^{-1} \in Z(D(L))$, hence by hypothesis $\alpha\alpha_1^{-1} = ss_1^{-1}$ where s, $s_1 \in T$. Therefore

$$s_1 tu_1^{m_1} \cdots u_r^{m_r} = st_1 u_1^{n_1} \cdots u_r^{n_r} \ (*)$$

Decompose $s_1 t$ and st_1 as a product of irreducible semi-invariants, which belong to T because T is saturated. By the unique decomposition, $(m_1, ..., m_r) = (n_1, ..., n_r)$, a contradiction].

Suppose that

$$\sum_{m=(m_1, ..., m_r)} a_m u_1^{m_1} \cdots u_r^{m_r} = 0, \qquad \text{where } a_m \in A.$$

Write $a_m = \sum_{\lambda_m} t_{\lambda_m}$, where $t_{\lambda_m} \in T$. Then

$$\sum_{m, \lambda_m} t_{\lambda_m} u_1^{m_1} \cdots u_r^{m_r} = 0.$$

By the first part of proof, if $m \neq m'$, $t_{\lambda_m} u_1^{m_1} \cdots u_r^{m_r}$ and $t_{\lambda_{m'}} u_1{}^{m'_1} \cdots u_r^{m'_r}$ have a different weight. Therefore, for all m

$$\sum_{\lambda_m} t_{\lambda_m} u_1^{m_1} \cdots u_r^{m_r} = 0,$$

i.e., $a_m u_1^{m_1} \cdots u_r^{m_r} = 0$, thus $a_m = 0$ for all m. ∎

Remark 5. There always exists a smallest set T satisfying the conditions of Proposition 4.

Proof. Let Σ be the set of all saturated, multiplicatively closed subsets S of E such that $Z(D(L)) = Q(S)$ and let T be the intersection of all these subsets. Clearly T is saturated and multiplicatively closed. Take any nonzero $z \in Z(D(L))$. By [4, p. 329] there exist relatively prime semi-invariants $u_0, v_0 \in E$ such that $z = u_0 v_0^{-1}$. In particular, $E \in \Sigma$. It suffices to show that $u_0, v_0 \in T$. Now, take any $S \in \Sigma$. Then $z = uv^{-1}$ for some $u, v \in S$. This implies $u_0 v = u v_0$. Hence u_0 divides u in the factorial ring $Sz(U(L))$, as u_0 and v_0 are relatively prime. Since $u \in S$ and S is saturated, it follows that $u_0 \in S$. Similarly, $v_0 \in S$. Because S is an arbitrary element of Σ, we may conclude that $u_0, v_0 \in T$.

Let T be a multiplicatively closed subset of E containing 1. Denote by Λ_T the set of weights of the elements of T. Clearly, Λ_T is a submonoid of $\Lambda(L)$.

THEOREM 6. *Let T be a saturated multiplicatively closed subset of E such that $Z(D(L)) = Q(T)$. Denote by A the subalgebra of $Sz(U(L))$ generated by T. Then the following hold*:

(1) *$U(L)$ has at most a finite number of nonassociated irreducible semi-invariants $u_1, ..., u_r$ not in T. Let $\lambda_1, ..., \lambda_r \in \Lambda(L)$ be their weights and let Λ' be the submonoid of $\Lambda(L)$ generated by these.*

(2) *$Sz(U(L)) = A[u_1, ..., u_r]$, a polynomial ring. So A is factorial.*

(3) *$Z(U(L)) \subset \bigcup_{\lambda \in \Lambda^u(L)} U(L)_\lambda \subset T \cup \{0\}$. In particular, $Sz^u(U(L)) \subset A$ and $\Lambda^u(L) \subset \Lambda_T$.*

(4) *$U(L)_{\lambda_i} = Z(U(L))\, u_i$, $i = 1, ..., r$.*

(5) *Each u_i is a semi-invariant for every derivation D of L.*

(6) *$\lambda_1, ..., \lambda_r$ are irreducible in $\Lambda(L)$.*

(7) *$\Lambda(L) = \Lambda_T \oplus \Lambda'$ and Λ' is a free abelian monoid.*

(8) *$\lambda_1, ..., \lambda_r$ form a $\mathbb{Z}$-basis of the subgroup $\langle \Lambda' \rangle$ of $\Lambda_D(L)$ generated by Λ'. Moreover, $\Lambda_D(L) = \langle \Lambda_T \rangle \oplus \langle \Lambda' \rangle$.*

Proof. (1) By Proposition 4, nonassociated irreducible semi-invariants not in T are algebraically independent over A and hence over k. Since semi-invariants commute and since the Gelfand–Kirillov dimension of $U(L)$ is finite [10, p. 78], there can only be a finite number of such semi-invariants, say $u_1, ..., u_r$. It is easy to verify there is at least one if $T \neq E$.

(2) By [4, Proposition 1.5] each nonzero semi-invariant u can be written uniquely as

$$u = t u_1^{m_1} \cdots u_r^{m_r}, \qquad \text{where} \quad t \in T, m_i \in \mathbb{N}.$$

It follows that $Sz(U(L)) = A[u_1, ..., u_r]$.

(3) *Let* $0 \neq a \in U(L)_\lambda$, $\lambda \in \Lambda^u(L)$. Take $0 \neq b \in U(L)_{-\lambda}$. Then $ab \in Z(U(L)) \subset Z(D(L)) = Q(T)$.

So $ab = st^{-1}$ for some s, $t \in T$. Thus $tab = s \in T$ and hence $a \in T$ since T is saturated. Therefore $U(L)_\lambda \subset T \cup \{0\}$.

(4) Take $0 \neq v \in U(L)_{\lambda_i}$. Then, $vu_i^{-1} \in Z(D(L))$ so $vu_i^{-1} = t't^{-1}$ for some t, $t' \in T$. Hence $tv = t'u_i$. Now u_i is prime in the factorial ring $Sz(U(L))$ and does not divide t (otherwise $u_i \in T$ as T is saturated). Consequently u_i divides v, i.e., $v = cu_i$ for some $c \in E$. But $c = vu_i^{-1} \in Z(D(L)) \cap U(L) = Z(U(L))$.

(5) Let D be a derivation of L. By [15, p. 265] D maps $U(L)_\lambda$ into itself. So $Du_i = cu_i$ for some $c \in Z(U(L))$. On the other hand $\deg(Du_i) \leqslant \deg u_i$[5, 2.5.9.] where deg denotes the usual degree function in $U(L)$. This forces $c \in k$.

(6) First $\lambda_i \neq 0$ since $u_i \notin T$ and $Z(U(L)) \backslash \{0\} \subset T$. Next suppose $\lambda_i = \alpha + \beta$ for some α, $\beta \in \Lambda(L)$. Take $0 \neq v \in U(L)_\alpha$ and $0 \neq w \in U(L)_\beta$. Then $vw \in U(L)_{\lambda_i}$ and thus $vw = cu_i$ for some nonzero $c \in Z(U(L))$. In $Sz(U(L))$ u_i is prime and divides vw. Therefore u_i divides v (or w), i.e., $v = u_i u$ for some nonzero $u \in U(L)_\mu$. In particular, $\alpha = \lambda_i + \mu$. It follows that $\beta + \mu = 0$ and thus β is a unit in $\Lambda(L)$.

(7) It clearly suffices to show that every $\lambda \in \Lambda(L)$ can be written uniquely as

$$\lambda = \lambda_T + \sum_{i=1}^{r} m_i \lambda_i,$$

where $\lambda_T \in \Lambda_T$ and each m_i is a natural number.

Let $\lambda \in \Lambda(L)$ and take a nonzero $u \in U(L)_\lambda$. As in (2) we have

$$u = t u_1^{m_1} \cdots u_r^{m_r},$$

where $t \in T$ and each m_i is a natural number.

Thus

$$\lambda = \tau + \sum_{i=1}^{r} m_i \lambda_i,$$

where $\tau \in \Lambda_T$ is the weight of t.

Now, suppose that we also have

$$\lambda = \tau' + \sum_{i=1}^{r} n_i \lambda_i,$$

where $\tau' \in \Lambda_T$ and $n_i \in \mathbb{N}$.

Pick $t' \in T$ with weight τ', then clearly $t u_1^{m_1} \cdots u_r^{m_r}$ and $t' u_1^{n_1} \cdots u_r^{n_r}$ have the same weight. As in the proof of Proposition 4, we may conclude that $m_i = n_i$ for all i, and hence also $\tau = \tau'$.

(8) This follows from (7). ∎

Theorem 6 can be adapted and extended to graded domains [22].

LEMMA 7. *Let L be a nonabelian Lie algebra, S a multiplicatively closed subset of E, then the Jacobson radical $J(U(L)_S) = 0$.*

Proof. We may clearly assume that S is saturated. Suppose that $J(U(L)_S) \neq 0$. Let $0 \neq a \in J(U(L)_S)$. Since L is nonabelian, we can find an $x \in U(L) \backslash Sz(U(L))$. Thus $1 + a$ and $1 + xa$ are invertible in $U(L)_S$. Therefore

$$1 + a = st^{-1} \qquad \text{and} \qquad 1 + xa = s_1 t_1^{-1},$$

where s, s_1, t, $t_1 \in S$. Also $s \neq t$ since $a \neq 0$. Thus

$$x(st^{-1} - 1) = s_1 t_1^{-1} - 1$$

and multiplying by tt_1 yields

$$x((s-t)\,t_1) = (s_1 - t_1)\,t \in Sz(U(L))$$

and

$$0 \neq (s-t)\,t_1 \in Sz(U(L)).$$

By [11, p. 398] $x \in Sz(U(L))$, a contradiction. ∎

LEMMA 8. *Let A be a left primitive ring, S a multiplicatively closed subset of A consisting of nonzero normalizing elements. Then the localization A_S is also left primitive.*

Proof. If $s \in S$, then s is not a zero divisor of A because A is prime. Since A is left primitive, A has a maximal left ideal I containing no nonzero two-sided ideal. But then $I \cap S = \varnothing$ because S consists of normalizing elements. It follows that $A_S I$ is a maximal left ideal of A_S containing no nonzero two-sided ideal. ∎

COROLLARY 9. *Let S, T be multiplicatively closed sets with $S \subset T \subset E$. If $U(L)_S$ is primitive, then $U(L)_T$ is primitive.*

Proof. Since semi-invariants commute, $U(L)_T = (U(L)_S)_T$. The result follows immediately from Lemma 8. ∎

PROPOSITION 10. *Let K be a subfield of $Z(D(L))$ with $k \subset K$. Put $U(L)\,K = \{\sum_i u_i a_i \mid u_i \in U(L),\ a_i \in K\}$, a subalgebra of $D(L)$. Then $U(L)\,K$ is primitive if and only if $Z(D(L))$ is algebraic over K.*

Proof. Since $U(L) \otimes_k K \cong U(L \otimes_k K)$, there exists a surjective homomorphism ϕ: $U(L \otimes_k K) \to U(L)\,K$: $\sum_i u_i \otimes a_i \to \sum_i u_i a_i$.

Then $U(L \otimes_k K)/P \cong U(L)\,K$ where $P = \ker \phi$. Hence $U(L)\,K$ is primitive precisely when P is primitive and this happens if and only if the center of the division ring of quotients of $U(L \otimes_k K)/P$ is algebraic over K [8, 21, p. 39]. But this is clearly equivalent to saying that $Z(D(L))$ is algebraic over K. ∎

COROLLARY 11. *$U(L)\,Z(D(L))$, the central closure of $U(L)$, is always primitive.*

THEOREM 12. *Let L be a nonabelian Lie algebra and T a saturated, multiplicatively closed subset of E. Consider the following conditions:*

(1) *$U(L)\,Q(T)$ is primitive and T is weakly additively closed;*

(2) *$Z(D(L))$ is algebraic over $Q(T)$ and T is weakly additively closed;*

(3) *$Z(D(L)) = Q(T)$;*

(4) *$U(L)$ contains at most a finite number of nonassociated irreducible semi-invariants $u_1, \ldots, u_r$ not contained in T;*

(5) *$U(L)$ contains a semi-invariant e such that e belongs to each nonzero prime ideal P of $U(L)$ such that $P \cap T = \emptyset$;*

(6) *$U(L)_T$ contains only a finite number of height one prime ideals;*

(7) *$U(L)_T$ is primitive.*

Then $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7)$ *holds.*

Proof. $(1) \Leftrightarrow (2)$ by Proposition 10.

$(3) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (3)$ Let $0 \neq z \in Z(D(L))$ be algebraic over $Q(T)$. There exist relatively prime semi-invariants $u, v \in U(L)_\lambda$ for some $\lambda \in \Lambda(L)$ such that $z = uv^{-1}$ [4, p. 329].

Suppose that

$$\sum_{i=0}^{n} a_i z^i = 0,$$

where we may assume that each $a_i \in T \cup \{0\}$, $a_n \neq 0$ and $a_0 \neq 0$. Then

$$\sum_{i=0}^{n} a_i u^i v^{n-i} = 0.$$

This implies that v divides $a_n u^n$ in $Sz(U(L))$ and thus also a_n as u and v are relatively prime. So there is a $v' \in E$ such that $vv' = a_n \in T$. Hence $v \in T$ as T is saturated. Similarly $u \in T$. Therefore $z \in Q(T)$.

$(3) \Rightarrow (4)$ is shown in Theorem 6.

$(4) \Rightarrow (5)$ We may assume that $T \neq E$. Let $e = u_1 \cdots u_r$ and let P be a nonzero prime ideal of $U(L)$ with $P \cap T = \emptyset$. Then P contains a nonzero semi-invariant u which can be written

$$u = u_T u_1^{m_1} \cdots u_r^{m_r}, \text{ where } u_T \in T.$$

Since $u_T, u_1, \ldots, u_r$ are normalizing elements and P is prime with $P \cap T = \emptyset$, if follows that $u_i \in P$ for some i. Hence $e \in P$.

$(5) \Rightarrow (6)$ By (5) each nonzero prime ideal of $U(L)_T$ contains the nonzero ideal $U(L)_T e$. Since $U(L)_T$ is Noetherian, it follows that $U(L)_T$ has only a finite number of height one prime ideals [5, p. 3.1.10].

$(6) \Rightarrow (7)$ Each nonzero prime ideal of $U(L)_T$ contains a height one prime ideal because $U(L)_T$ satisfies the descending chain condition on nonzero prime ideals. By (6) the intersection of the nonzero prime ideals of $U(L)_T$ is nonzero and hence the same holds for the nonzero primitive ideals of $U(L)_T$. On the other hand, $J(U(L)_T)=0$ by Lemma 7. Therefore $U(L)_T$ is primitive. ∎

Note that $(1) \Rightarrow (7)$ is a direct consequence of Lemma 8 and the fact that $U(L)_T = (U(L)\, Q(T))_T$.

We do not know whether the implication $(7) \Rightarrow (1)$ always holds. The next result provides a few cases when it does.

THEOREM 13. *Let be a nonabelian Lie-algebra.*

(1) *Let S be a multiplicatively closed subset of $Z(U(L))\backslash\{0\}$ such that $T=\mathrm{Sat}(S)$ is weakly additively closed. Then*

$$U(L)_S = U(L)_T \textit{ is primitive} \Leftrightarrow Z(D(L)) = Q(T).$$

(2) *Let S be a subalgebra of $Z(U(L))$ and $T=\mathrm{Sat}(S\backslash\{0\})$. Then*

$$\begin{aligned} U(L)_S = U(L)_T \textit{ is primitive} &\Leftrightarrow Z(D(L)) \textit{ is algebraic over } Q(S) \\ &\Leftrightarrow Z(D(L)) = Q(T). \end{aligned}$$

In particular, if $U(L)_S$ is primitive, then T is weakly additively closed.

If S is as in (1) or (2) and if $U(L)_S$ is primitive, then $U(L)_S = U(L)_{Z(U(L))}$; i.e., S and $Z(U(L))$ have the same saturation.

Proof. (1) If $s \in S$, then $s^{-1} = 1 \cdot s^{-1} \in Q(T)$. Hence $U(L)_S \subset U(L)\, Q(T) \subset U(L)_T = U(L)_S$ Therefore $U(L)_S = U(L)\, Q(T)$ and the result follows from Theorem 12.

(2) Of course, $U(L)_S$ denotes the localization of $U(L)$ at the nonzero elements of S. If $U(L)_S = U(L)\, Q(S)$ is primitive, then by Proposition 10, $Z(D(L))$ is algebraic over $Q(S)$. By the proof of Theorem 12, step $(2) \Rightarrow (3)$, we have $Z(D(L)) = Q(T)$. In particular, T is weakly additively closed. Conversely, if $Z(D(L)) = Q(T)$, then $U(L)_S = U(L)_T$ is primitive by Theorem 12.

Finally, if S is as in (1) or (2) and if $U(L)_S$ is primitive, then by Theorem 6, $S\backslash\{0\} \subset Z(U(L))\backslash\{0\} \subset T$. Therefore $U(L)_S \subset U(L)_{Z(U(L))} \subset U(L)_T = U(L)_S$, hence equality holds. ∎

COROLLARY 14. *Let L be a nonabelian Lie algebra. Then*

$$U(L)_{Z(U(L))} \textit{ is primitive if and only if } Z(D(L)) = Q(Z(U(L))).$$

Suppose $U(L)_{Z(U(L))}$ is primitive and $Sz(U(L)) \neq Sz^u(U(L))$. Then:

(1) *$Sz(U(L)) = Sz^u(U(L))[u_1, ..., u_r]$, a polynomial ring, where $u_1, ..., u_r$ are the nonassociated irreducible semi-invariants not in $Sz^u(U(L))$. Let $\lambda_1, ..., \lambda_r \in \Lambda(L)$ be their weights and let Λ' be the submonoid of $\Lambda(L)$ generated by these.*

(2) *$\Lambda(L) = \Lambda^u(L) \oplus \Lambda'$ and Λ' is a free abelian monoid. In particular, $\Lambda(L)$ is a finitely generated, factorial monoid.*

(3) *Let $\mu_1, ..., \mu_t \in \Lambda^u(L)$ be a $\mathbb{Z}$-basis of $\Lambda^u(L)$. Then $\mu_1, ..., \mu_t, \lambda_1, ..., \lambda_r$ is a $\mathbb{Z}$-basis of $\Lambda_D(L)$ contained in $\Lambda(L)$.*

(4) *$e = u_1 \cdots u_r$ is a semi-invariant of $U(L)$ for all automorphisms and anti-automorphisms of $U(L)$.*

(5) *Let P be a nonzero prime ideal of $U(L)$. Then P contains either e or a nonzero central element of $U(L)$.*

Proof. By Theorem 13 $U(L)_{Z(U(L))}$ is primitive precisely when $Z(D(L)) = Q(T)$ where $T = \mathrm{Sat}(Z(U(L))\setminus\{0\}) = \bigcup_{\lambda \in \Lambda^u(L)} U(L)_\lambda \setminus \{0\}$. Let $z \in Q(T)$, i.e., $z = uv^{-1}$ where u, $v \in U(L)_\lambda$ for some $\lambda \in \Lambda^u(L)$. Choose a nonzero $w \in U(L)_{-\lambda}$. Then $z = (uw)(vw)^{-1} \in Q(Z(U(L)))$. So $Q(T) = Q(Z(U(L)))$. Now let us assume that $U(L)_{Z(U(L))}(= (U(L)_T)$ is primitive.

(1) This follows at once from (2) of Theorem 6 since $Sz^u(U(L))$ is the subring of $Sz(U(L))$ generated by T.

(2) The first part is clear by (7) of Theorem 6 as $\Lambda_T = \Lambda^u(L)$. Also, each $\lambda \in \Lambda(L)$ has a unique decomposition

$$\lambda = \mu + \sum_{i=1}^{r} m_i \lambda_i,$$

where $\mu \in \Lambda^u(L)$ is a unit of $\Lambda(L)$, $m_i \in \mathbb{N}$ and $\lambda_1, ..., \lambda_r$ are irreducible in $\Lambda(L)$ ((6) of Theorem 6). Hence $\Lambda(L)$ is a factorial monoid [9, p. 136]. It is also finitely generated since both $\Lambda^u(L)$ and Λ' are finitely generated monoids.

(3) This follows from (8) of Theorem 6 and the fact that $\Lambda_T = \Lambda^u(L)$.

(4) Let α be either an automorphism or an anti-isomorphism of $U(L)$. Then α maps $Sz^u(U(L))$ onto itself and $\alpha(u_i)$ is an irreducible semi-invariant not contained in $Sz^u(U(L))$. Hence α permutes the elements of the set $\{k^*u_1, ..., k^*u_r\}$. Therefore $\alpha(e) = ae$ for a suitable $a \in k^*$.

(5) Let P be a nonzero prime ideal of $U(L)$. If $P \cap T = \varnothing$ then $e \in P$ by the proof (4) $\Rightarrow$ (5) of Theorem 12. Otherwise P contains an element $t \in T$. Then there is a suitable $t' \in T$ such that $z = tt' \in Z(U(L))$. So, z is a nonzero central element contained in P. ∎

Remarks 15. (1) The following example shows that $Z(U(L))$ need not be factorial. Let L be the Lie algebra over k with basis x, y, z, w, t with nonvanishing brackets $[t, x] = x$, $[r, y] = y$, $[t, z] = -z$, $[t, w] = -w$. Then $\Lambda(L) = \Lambda^u(L)$ is an infinite cyclic group, $Sz(U(L)) = k[x, y, z, w]$ but

$$Z(U(L)) = k[xz, xw, yz, yw]$$

is not factorial since $(xz)(yw) = (xw)(yz)$ and xz, yw, xw, yz are irreducible in $Z(U(L))$.

Note that $U(L)_{Z(U(L))}$ is simple by Proposition 3.

(2) If S is a subalgebra of $Z(U(L))$ such that $U(L)_S$ is primitive, then $Z(D(L))$ is algebraic over $Q(S)$ by Theorem 13. This does not imply however that $Z(D(L)) = Q(S)$. For let L be the Lie algebra over k with basis x, y, z such that $[x, y] = y$, $[x, z] = -z$ and $[y, z] = 0$. Then $\Lambda(L) = \Lambda^u(L)$ is an infinite cyclic group, $Sz(U(L)) = k[y, z]$ and $Z(U(L)) = k[yz]$. By Proposition 3, $U(L)_{Z(U(L))}$ is simple. In particular $Z(D(L)) = Q(Z(U(L)))$. Put $S = k[(yz)^2]$ then $U(L)_S = U(L)_{Z(U(L))}$ for if $f = f_0 + yzf_1 \in Z(U(L))$ where $f_0, f_1 \in k[(yz)^2]$, then $fg \in S$ where $g = f_0 - yzf_1$. In particular, $U(L)_S$ is primitive and thus $Z(D(L))$ is algebraic over $Q(S)$ by Theorem 13. But clearly $Q(S) \neq Z(D(L))$.

(3) We know that the monoid $\Lambda(L)$ need not be finitely generated [6, p. 321]. Also, $\Lambda(L)$ is not factorial in general. Indeed, let L be the Lie algebra over k with basis x, y, z such that $[y, z] = 0$, $[x, y] = py$ and $[x, z] = qz$ where p, q are relatively prime natural numbers. Then the monoid $\Lambda(L)$ is generated by the weights λ, μ vanishing on y and z and for which $\lambda(x) = p$ and $\mu(x) = q$.

Clearly, $\Lambda(L)$ is not factorial since the relation $q\lambda = p\mu$ holds between the irreducible elements λ and μ.

Finally, we want to establish the necessary and sufficient condition on the Lie algebra L in order for the localization $U(L)_R$ to be primitive where $R = U(Z(L))$, i.e., the enveloping algebra of the center $Z(L)$ of L. Note that $R \subset Z(U(L))$ and R can be identified with the symmetric algebra $S(Z(L))$ since $Z(L)$ is commutative. Hence R may be considered as a subalgebra of both $U(L)$ and $S(L)$, the symmetric algebra of L. Let $K(L)$ (resp. $K(Z(L))$) be the quotient field of $S(L)$ (resp. $S(Z(L))$) and $K(L)^I$ the subfield of invariants of $K(L)$ under the action of ad L. On the other hand, for each $f \in L^*$ we denote by $L[f]$ the collection of elements $x \in L$ such that $f(Ex) = 0$ for all $E \in H$, H being the algebraic hull of ad L in End L. $L[f]$ is a Lie subalgebra of L containing $Z(L)$. Let $x_1, ..., x_n$ be a basis for L and $E_1, ..., E_m$ one for H, then we have the following formula for the degree of transcendence over k of $K(L)^I$ [18, p. 492]

$$\begin{aligned} \operatorname{tr\,deg}_k(K(L)^I) &= \dim L - \operatorname{rank}_{K(L)}((E_i x_j)_{ij}) \\ &= \min_{f \in L^*} \dim L[f]. \end{aligned}$$

Furthermore, let $s: S(L) \to U(L)$ be the symmetrization map, i.e., the canonical linear bijection which maps each product $y_1 \cdots y_q$, $y_i \in L$, into $(1/q!) \sum_p y_{p(1)} \cdots y_{p(q)}$, where p ranges over all permutations of $\{1, ..., q\}$. s is known to commute with each derivation of L [5, 2.4.9] and hence maps $S(L)_\lambda$ onto $U(L)_\lambda$ and also $Sz(S(L))$ onto $Sz(U(L))$. (of course, $S(L)_\lambda$ is the set of all $y \in S(L)$ such that $\operatorname{ad} x(y) = \lambda(x) y$ for all $x \in L$ and $\mathrm{Sz}(S(L)) = \bigoplus_\lambda S(L)_\lambda$). Also, $s(ax) = as(x)$ for all $a \in R$ and $x \in S(L)$. As a result s may be considered as an isomorphism of R-modules.

PROPOSITION 16. *The following conditions are equivalent*:

(1) $U(L)_R$ *is primitive where* $R = U(Z(L))$

(2) $Z(D(L))$ *is algebraic over* $D(Z(L))$

(3) $Z(D(L)) = D(Z(L))$

(4) $K(L)^I = K(Z(L))$

(5) $L[f] = Z(L)$ *for some* $f \in L^*$ (*i.e.*, $\mathit{rank}(E_i x_j) = \dim L - \dim Z(L)$).

Moreover, these conditions imply that

(a) $Z(U(L)) = U(Z(L))$

(b) $U(L)$ *admits at most a finite number of nonassociated irreducible semi-invariants* $u_1, ..., u_r$ *not contained in* $Z(U(L))$

(c) $$Sz(U(L)) = Z(U(L))[u_1, ..., u_r] = k[c_1, ..., c_q; u_1, ..., u_r]$$

a polynomial algebra over k, *where* $c_1, ..., c_q$ *is a basis of* $Z(L)$.

Proof. The equivalence of (1), (2), and (3) is a direct consequence of (2) of Theorem 13 since $R\backslash\{0\}$ is saturated [13, p. 1271]. On the other hand, (a), (b), and (c) follow from (1), (2), and (3) of Theorem 6. The equivalence of (3), (4), and (5) was the subject of [18, Proposition 4]. We give a new proof for the implication (4) $\Rightarrow$ (3) since the one given in [18] does not hold in general (it does however if $Z(L) = 0$ being precisely the case in which it was used in [18]). So, let $z \in Z(D(L))$. Then $z = uv^{-1}$ for some u, $v \in U(L)_\lambda$, $v \neq 0$. Choose x, $y \in S(L)_\lambda$ such that $u = s(x)$, $v = s(y)$. Clearly $xy^{-1} \in K(L)^I = K(Z(L))$. Hence $xy^{-1} = ab^{-1}$ for some a, $b \in S(Z(L)) = R$, $b \neq 0$ and thus $bx = ay$. Then $bu = bs(x) = s(bx) = s(ay) = as(y) = av$. Therefore $z = uv^{-1} = ab^{-1} \in D(Z(L))$. ∎

EXAMPLES. (1) The Lie algebras L for which $L(f) = Z(L)$ for some $f \in L^*$. (We recall that $L(f) = \{x \in L \mid f([x, y]) = 0 \text{ for all } y \in L\}$ and that $Z(L) \subset L[f] \subset L(f)$). If in addition $Z(L) = 0$, then L is called Frobenius (see, e.g., [7; 18, p. 497; 19]). On the other hand, in the nilpotent case these Lie algebras are precisely the Lie algebras of simply connected nilpotent Lie groups having square integrable representations [16, Theorem 3], e.g., the $2n+1$ dimensional Heisenberg Lie algebra with basis $x_1, ..., x_n, y_1, ..., y_n, z$ with nonvanishing brackets $[x_i, y_i] = z$ i: $1, ..., n$.

(2) Let M be a finite dimensional Frobenius Lie algebra over k having a nonzero semi-invariant $u \in M$. (such a semi-invariant exists for instance in the completely solvable case) Let L be the centralizer of u in M. Clearly, L is an ideal of M of codimension one. Then $Z(D(L)) = k(u) = D(Z(L))$ by [4, Theorem 4.5]. In fact one can verify directly that L satisfies the condition of (1).

LEMMA 17. *If* $Z(D(L)) = D(Z(L))$ *then* $s\colon Sz(S(L)) \to Sz(U(L))$ *is an algebra isomorphism.*

Proof. First we introduce an increasing filtration in $U(L)$ other than the usual one. Let $x_1, ..., x_p, x_{p+1}, ..., x_n$ be a basis of L such that $x_{p+1}, ..., x_n$ is a basis of $Z(L)$. Each element of $S(L)$ can be considered as a polynomial in $x_1, ..., x_p$ with coefficients in $S(Z(L)) = R$. Clearly $S(L)$ is the direct sum of the subspaces S^m of polynomials homogeneous of degree m in $x_1, ..., x_p$. Therefore $U(L)$ is the direct sum of the subspaces U^m, where $U^m = s(S^m)$. Next put $U_q = \oplus_{m \leq q} U^m$. In particular $U_0 = U^0 = R = S^0$. Then it is easy to verify that the subspaces U_q form an increasing filtration in $U(L)$ and the associated graded algebra $\mathrm{gr}(U(L))$ is isomorphic to $R[X_1, ..., X_p] \cong S(L)$. The elements $u \in U_q \backslash U_{q-1}$ are said to be of degree q and $[u] = u \bmod U_{q-1}$ is called the leading term of u. For all nonzero u, $v \in U(L)$ we have $[uv] = [u][v]$. Furthermore, if $y = y_m + \cdots + y_0$, $y_m \neq 0$, is the decomposition of $y \in S(L)$ into homogeneous components $y_i \in S^i$ then it follows from the definition of s that $[s(y)] = y_m$. Next, let $y, z \in S(L)$ be nonzero semi-invariants of $S(L)$ with weights λ and μ. Since $s\colon Sz(S(L)) \to Sz(U(L))$ is a linear isomorphism, it suffices to show that $s(yz) = s(y)\, s(z)$. Clearly, $y \in S(L)_\lambda$, $z \in S(L)_\mu$ and $yz \in S(L)_{\lambda+\mu}$. Hence $s(y) \in U(L)_\lambda$, $s(z) \in U(L)_\mu$ and $s(yz) \in U(L)_{\lambda+\mu}$. Also $s(y)\, s(z) \in U(L)_{\lambda+\mu}$. Therefore, $s(y)\, s(z)\, s(yz)^{-1} \in Z(D(L)) = D(Z(L))$ and thus can be written as ab^{-1} for some nonzero a, $b \in R$. This implies that $as(yz) = bs(y)\, s(z)$. Taking leading terms we obtain $a[s(yz)] = [a][s(yz)] = [as(yz)] = [bs(y)\, s(z)] = [b][s(y)\, s(z)] = b[s(y)\, s(z)]$. But $[s(yz)] = [s(y)\, s(z)]$. Consequently, $a = b$ and so $s(yz) = s(y)\, s(z)$. ∎

Next, we want to demonstrate that in case $Z(D(L)) = D(Z(L))$ the irreducible semi-invariants are very easy to construct. Because of Lemma 17 it suffices to do so in $S(L)$. For this purpose we recall the special semi-invariants introduced in [4, Theorem 2.2]: let $x_1, ..., x_n$ be a basis for L and $E_1, ..., E_m$ one for H, the algebraic hull of ad L. Let d be the (determinantal) rank of the $m \times n$ matrix $A = (E_i x_j)$ with entries in $S(L)$ and denote by $\Delta_q(L)$, $1 \leqslant q \leqslant d$, the greatest common divisor of all q-rowed minors of A. Then $\Delta_q(L)$, which is well defined up to a nonzero scalar, is a nonzero semi-invariant under the action of Aut L (and hence of Der L).

THEOREM 18. *Suppose $U(L)_R$ is primitive where $R = U(Z(L))$. Let $u_1, ..., u_r$ be the nonassociated irreducible semi-invariants of $U(L)$ not contained in R and put $v_1 = s^{-1}(u_1), ..., v_r = s^{-1}(u_r)$. Then we have*

(1) *$v_1, ..., v_r$ are the only nonassociated irreducible semi-invariants of $S(L)$ not contained in R. Moreover,*

$$Sz(S(L)) = R[v_1, ..., v_r]$$

(2) *$v_1, ..., v_r$ are precisely the irreducible factors of $\Delta_d(L)$ not contained in R, where $d = \dim L - \dim Z(L)$.*

(3) *$Sz(U(L)) = Z(U(L))$ if and only if $\Delta_d(L) \in R$.*

Proof. (1) follows directly from Proposition 16 and Lemma 17.

(2) Let $x_1, ..., x_d, x_{d+1}, ..., x_n$ be a basis of L such that $x_{d+1}, ..., x_n$ is a basis of $Z(L)$ and let $E_1, ..., E_m$ be a basis of H. Consider the $m \times n$ matrix $A = (E_i x_j)$. In view of Proposition 16 we know that $\operatorname{rank}(A) = \dim L - \dim Z(L) = d$.

Since ad L acts trivially on $Z(L)$ so does its algebraic hull H [3, p. 208]. Therefore $E_i x_j = 0$ for all i: $1, ..., m$ and j: $d+1, ..., n(*)$. Now take any irreducible semi-invariant v of $S(L)$ not contained in R. It suffices to show that v divides $\Delta_d(L)$. For this purpose let B be any $d \times d$ nonsingular submatrix of A. Clearly, we may assume that $B = (E_i x_j)$ where i, j: $1, ..., d$. Since v is a semi-invariant for ad L, it is also one for H, H being the smallest algebraic Lie subalgebra of Der L containing ad L [3, p. 208] (or otherwise note that property (5) of Theorem 6 with $T = R$ also holds for the irreducible semi-invariants of $S(L)$ not contained in R). Hence there exists a linear function $\mu \in H^*$ such that

$$E_i v = \mu(E_i)\, v \qquad \text{for all} \quad i\colon 1, ..., m.$$

In particular,

$$\sum_{j=1}^{n} (E_i x_j) \frac{\partial v}{\partial x_j} = \mu(E_i)\, v, \qquad i\colon 1, ..., d.$$

Because of $(*)$,

$$\sum_{j=1}^{d} (E_i x_j) \frac{\partial v}{\partial x_j} = \mu(E_i)\, v, \qquad i\colon 1, ..., d.$$

By Cramer's rule,

$$\det B \frac{\partial v}{\partial x_j} = \sum_{i=1}^{d} B_{ij}\mu(E_i)\, v = \left(\sum_{i=1}^{d} B_{ij}\mu(E_i)\right) v, \qquad j: 1, ..., d.$$

As $v \notin R = S(Z(L))$ we may choose j such that $\partial v/\partial x_j$ is nonzero. Clearly, v divides $\det B(\partial v/\partial x_j)$ in $S(L)$, but does not divide $\partial v/\partial x_j$ as the latter has a lower degree than v. Since $S(L)$ is a UFD and v is irreducible, it follows that v divides $\det B$. We may conclude that v divides all d-rowed minors of the matrix A and hence also their greatest common divisor which is $\Delta_d(L)$.

(3) This is a direct consequence of (1) and (2), Lemma 17 and the fact that $Z(U(L)) = R$. ∎

References

1. D. F. Anderson, Graded Krull Domains, *Comm. Algebra* **7** (1979), 79–106.
2. D. D. Anderson and D. F. Anderson, Divisibility properties of graded domains, *Canad. J. Math.* **34** (1982), 196–215.
3. C. Chevalley, "Théorie des groupes de Lie," Vol. III, Hermann, Paris, 1968.
4. L. Delvaux, E. Nauwelaerts, and A. I. Ooms, On the semi-center of a universal enveloping algebra, *J. Algebra* **94** (1985), 324–346.
5. J. Dixmier, Enveloping algebras, in "North-Holland Mathematical Library," Vol. 14, North-Holland, Amsterdam, 1977.
6. J. Dixmier, M. Duflo, and M. Vergne, Sur la représentation coadjointe d'une algèbre de Lie, *Compositio Math.* **25**, No. 194 (1974), 309–323.
7. A. G. Elashvili, Frobenius Lie Algebras, II, *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk. Gruzin. SSR* **77** (1985), 127–137.
8. R. S. Irving and L. W. Small, On the characterization of primitive ideals in enveloping algebras, *Math. Z.* **173** (1980), 217–221.
9. N. Jacobson, "Basic Algebra I," Freeman, San Francisco, 1974.
10. G. R. Krause and T. H. Lenagan, Growth of algebras and Gelfand–Kirillov dimension, *in* "Research Notes in Mathematics," Vol. 116, Pitman, London, 1985.
11. L. Le Bruyn and A. I. Ooms, The semicenter of an enveloping algebra is factorial, *Proc. Amer. Math. Soc.* **93** (1985), 397–400.
12. M. P. Malliavin, Ultra produit d'algèbres de Lie, *in* "Lecture Notes in Mathematics," Vol. 924, pp. 157–166, Springer-Verlag, Berlin, 1982.
13. C. Moeglin, Factorialité dans les algèbres enveloppantes, *C. R. Acad. Sci. Paris A* **282** (1976), 1269–1272.
14. C. Moeglin, Idéaux bilatères dans les algèbres enveloppantes, *Bull. Soc. Math. France* **108** (1980), 143–186.
15. S. Montgomery, X-inner automorphisms of filtered algebras, *Proc. Amer. Math. Soc.* **83** (1981), 263–268.
16. C. C. Moore and J. A. Wolf, Square integrable representations of nilpotent groups, *Trans. Amer. Math. Soc.* **185** (1973), 445–462.

17. E. NAUWELAERTS AND A. I. OOMS, Weights of semi-invariants of the quotient division ring of an enveloping algebra, *Proc. Amer. Math. Soc.* **104** (1988), 13–19.
18. A. I. OOMS, On Lie algebras having a primitive universal enveloping algebra, *J. Algebra* **32** (1974), 488–500.
19. A. I. OOMS, On Frobenius Lie algebras, *Comm. Algebra* **8** (1980), 13–52.
20. R. RENTSCHLER AND M. VERGNE, Sur le semi-centre du corps enveloppant d'une algèbre de Lie, *Ann. Sci. École Norm. Sup (4)* **6** (1973), 389–405.
21. R. RENTSCHLER, Primitive ideals in enveloping algebras (General case), *Math. Surveys Monographs* **24** (1987); "Noetherian Rings and their Applications" (L. W. Small, Ed.), pp. 37–57, Amer. Math. Soc., Providence, R.I.
22. P. WAUTERS, Factorial domains and graded rings, *Comm. Algebra* **17** (1989), 827–836.